Threshold Driven Reentrant Phase Transitions on Multiplex Networks

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Overview

- Social influence is among the main mechanisms driving ideas, fads, and social movements to spread throughout a population
- Social relationships may constitute entirely different dyadic types, modelled as distinct layers in a multiplex network. For example, a real-world / online duplex
- Collective behaviour can be shown to result from threshold dynamics, where states of nodes in a multiplex depend on a fraction ϕ of neighbour influence

Model

- We use a multiplex network model, meaning every node is present in every layer, with a single overall state (susceptible or infected, vulnerable or adopted)
- We define the overlap γ_{ij} between each pair of layers i and j, with γ self-consistent
- We define the relative edge density δ_i, giving average degree $z = z_1 + \ldots + z_M$
- A multiplex with M layers allows $2^M - 1$ composite edge types, ignoring null edge
- $\mathbf{k}$ and $\mathbf{m}$ store the total number of neighbours k_j, and the number of infected neighbours m_j, of edge type j where $1 \leq j \leq 2^M - 1$. Store layer weights in $\mathbf{w}$

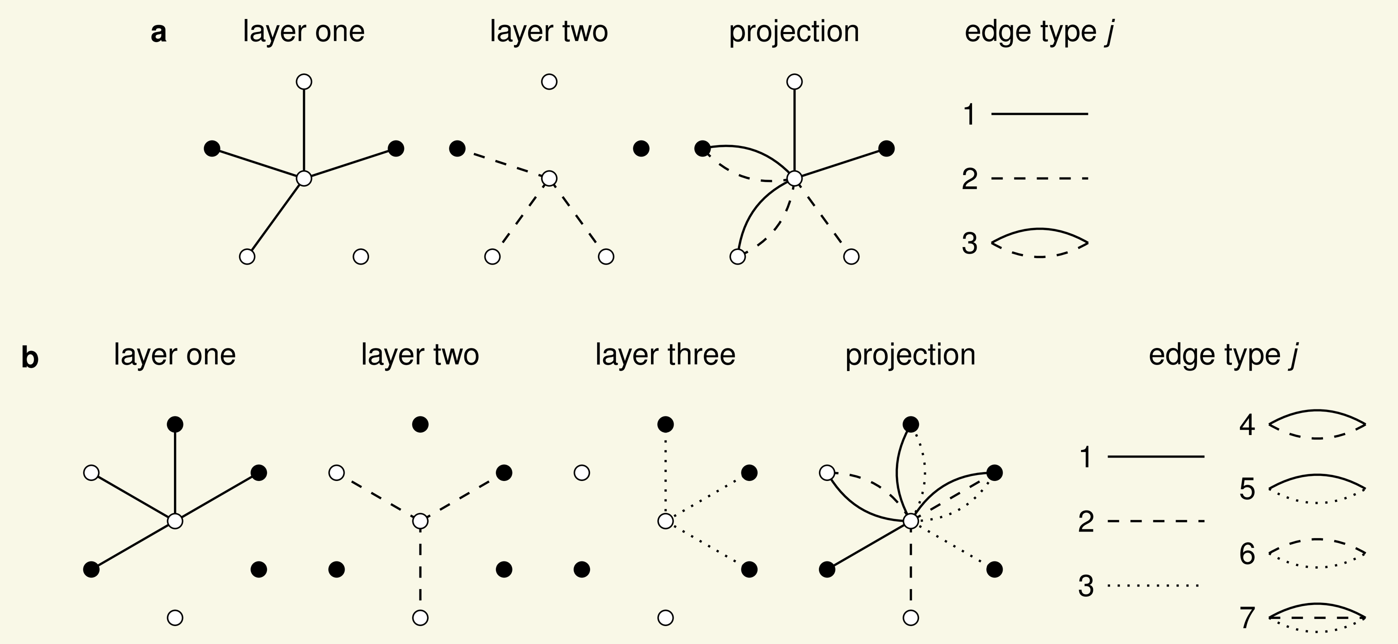

Figure: Example configurations in $M = 2$ and 3 layer multiplexes. An M layer multiplex induces $2^M - 1$ composite link types. The degree vectors for the projected network in **a** are $\mathbf{k} = (2, 1, 2)$ and $\mathbf{m} = (1, 0, 1)^T$. In plot **b**, we have $\mathbf{k} = (1, 1, 1, 1, 1, 0, 1)$ and $\mathbf{m} = (1, 0, 1, 0, 1, 0, 1)^T$.

Threshold rule

- In the most general formulation, we use a dynamic threshold rule with $F_{\mathbf{k},\mathbf{m}} = 1$ for adoption induced by neighbours, and $F_{\mathbf{k},\mathbf{m}} = p$ for spontaneous adoption.
- A number of adoption rules may be defined, based on whether the threshold is satisfied in one layer l, where $1 \leq l \leq M$, or all layers l. When layers are considered separately, a node may have a threshold ϕ_l for every edge type

	any layer	all layers	combined
$F_{\mathbf{k},\mathbf{m}}$	$\begin{cases} 1, & \exists l \text{ s.t. } m_l \geq \phi_l k_l \\ p, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & m_l \geq \phi_l k_l \,\forall l \\ p, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \mathbf{m}\cdot\mathbf{w} \geq \phi \mathbf{k}\cdot\mathbf{w} \\ p, & \text{otherwise} \end{cases}$

- In this work we focus on the combined definition of the threshold rule, although the following analytic formalism is applicable to all definitions of $F_{\mathbf{k},\mathbf{m}}$, where node behaviour is determined by a single threshold ϕ of all incoming stimuli

Analytical solution

- We build on a Master Equation formalism for edge-heterogenous networks in [1], extending analytical results first presented in [2]
- Let $\rho(t)$ be the total density of infected nodes, and $v_j(t)$ be the probability of a j type neighbour of a randomly selected node being infected, then

$$\dot{v}_j = g_j(v, t) - v_j \tag{1}$$
$$\dot{\rho} = h(v, t) - \rho \tag{2}$$

where

$$g_j(v, t) = f_t + (1 - f_t) \sum_{k,\mathbf{k}} \frac{k_j}{z_j} P(k) P(\mathbf{k}) \sum_{f=1} B_{k_j - 1, m_j}(v_j) \prod_{i \neq j} B_{k_i, m_i}(v_j) \tag{3}$$

and

$$h(v, t) = f_t + (1 - f_t) \sum_{k,\mathbf{k}} P(k) P(\mathbf{k}) \sum_{f=1} \prod_{j=1}^{n} B_{k_i, m_i}(v_j), \tag{4}$$

with

$$f_t = 1 - (1 - p) e^{-pt}. \tag{5}$$

Perturbation analysis

- Setting $p = 0$ in the reduced AMEs leads to $f_t = 0$, and we can write $g_j = g_j(v)$ and $h = h(v)$. Results from stability analysis allow us to derive a cascade condition

$$\mathcal{J}^*_{ji} = -\delta_{ji} + \frac{\partial g_j(v)}{\partial v_i}\bigg|_{v = v^*} = -\delta_{ji} + \sum_{k,\mathbf{k}} \frac{k_j}{z_j} (k_i - \delta_{ji}) P(k) P(\mathbf{k}) f(\mathbf{k}, \mathbf{e}_i) \tag{6}$$

- Since the system $\dot{v}_j = g_j(v) - v_j$ is closed, the stability of Equation 1 is determined by its stability at $v = 0$
- The eigenvalues λ_j of the Jacobian $\mathcal{J}^*$ are obtained by solving the characteristic equation $\det|\mathcal{J}^* - \lambda \mathbf{1}| = 0$. Results from stability analysis then state that the system is unstable when at least one eigenvalue is positive, or for any j

$$\mathcal{R}(\lambda_j) > 0 \tag{7}$$

- The results of stability analysis give us the outline of the non-zero phase in the figure below, where a single adopting seed node leads to a global cascade of adoption

Watts phase space

- In the Watts model [3], spontaneous adoption is disallowed, $p = 0$, and an initial seed of adopting nodes may trigger a global cascade via threshold dynamics
- Networks are generated using the multivariate configuration model, meaning the network is maximally random up to joint degree distribution
- Joint degree distribution is binomial, in individual layers as well as in overlap

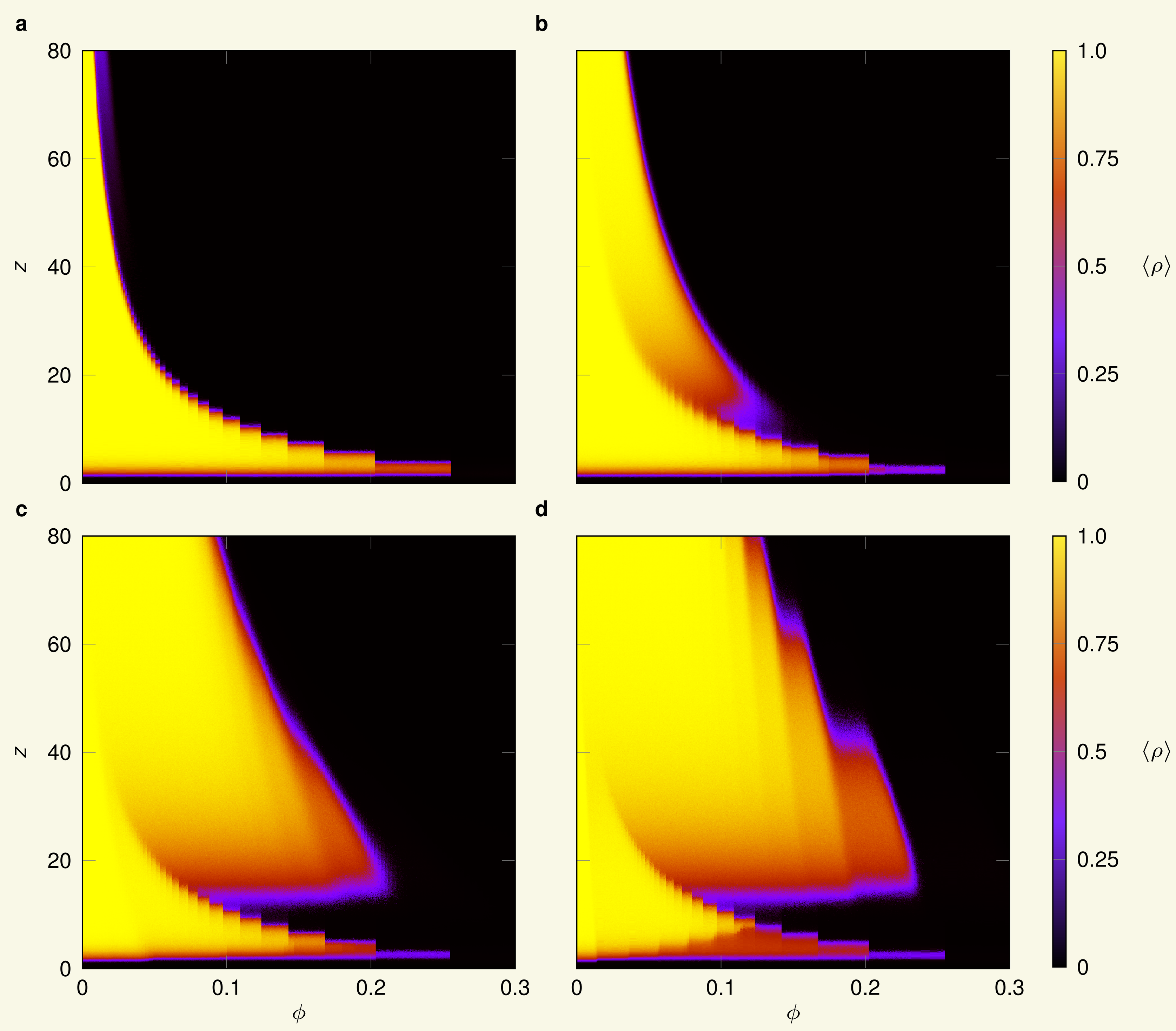

Figure: $M = 2$ layer multiplex with relative density $\delta = 0.1$, layer overlap $\gamma = 0$, varying the weight standard deviation σ. Intensity of the heat map is given by the expected size of the contagion cluster. Network size is $N = 10^4$ with average $\langle \rho \rangle$ found via 10^3 single node perturbations. Weight standard deviation σ is set to 0, 0.2, 0.6 and 0.8 respectively.

Conclusion and further work

- Observation of additional phase transitions when increasing average degree z for constant threshold ϕ. This is found in the case of sufficient relative edge density δ, and sufficient weight standard deviation σ
- Observation of partial cascades, where nodes occupy a global, but incomplete, fraction of a fully connected network
- Interplay between links from different layers can limit the size of global cascades
- Master Equation formalism accurately captures the behaviour observed in simulations on configuration model networks
- Future work will be to explore threshold driven contagion in temporal networks, building upon the results presented here

[1] S. Unicomb, G. Iñiguez, M. Karsai. Threshold driven contagion on weighted networks. *Sci Rep* **8**, 3094 (2018).
[2] J. P. Gleeson. High-accuracy approximation of binary-state dynamics on networks. *Phys Rev Lett* **107**, 068701 (2011).
[3] D. J. Watts. A simple model of global cascades on random networks. *Proc Natl Acad Sci USA* **99**, 5766–5771 (2002).

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